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3 (Sem-4 /CBCS) MAT HC 3

2021

MATHEMATICS

(Honours)

Paper : MAT-HC-4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. $1\times6=6$

(a) Give an example of an infinite non-commutative ring with unity.

(b) What is the characteristic of the ring $Z_3[i]$?

(c) Find all the idempotent elements of Z_{10}.

(d) Let $f(x)=4x^3+2x^2+x+3$ and $g(x)=3x^4+3x^3+3x^2+x+4$ where $f(x),\ g(x)\in Z_5[x]$. Compute $f(x).g(x)$.

(e) Show that the polynomial $x^5+9x^4+12x^2+6$ is irreducible over Q.

(f) Define an Euclidean Domain.

2. $2\times5=10$

(a) Prove that in a ring R, $a(-b)=(-a)b=-(ab)$ for all $a,\ b\in R$.

(b) Let A and B be two ideals of a ring R. Show that $AB\subseteq A\cap B$.

(c) If A is an ideal of a ring R and $1\in A$, then prove that $A=R$.

(d) If R is a commutative ring with unity and A is an ideal of R, show that R/A is a commutative ring with unity.

(e) Let $f(x)=x^3+2x+4$ and $g(x)=3x+2$ in $Z_5(x)$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. Answer ***any four*** parts : $6\times4=24$

(a) Let R be a commutative ring with unity and A be an ideal of R. Prove that R/A is a field if and only if A is a maximal ideal. 6

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(b) (i) Let R be a commutative ring and A be an ideal of it. Show that the set $\{r \in R \mid ra = 0, \text{ for all } a \in A\}$ is an ideal of R. 3

(ii) Prove that the characteristic of an integral domain is 0 or prime. 3

(c) Define a principal ideal domain. If F is a field, then show that $F(x)$ is a principal ideal domain. 1+5=6

(d) Let p be a prime and $f(x) \in Z(x)$ with $deg\ f(x) \geq 1$. Suppose $\overline{f(x)}$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $\overline{f(x)}$ is irreducible over over Z_p and $deg\ f(x) = deg\ \overline{f(x)}$, then prove that $f(x)$ is irreducible over Q. Is the converse true ? Justify your answer. 4+2=6

(e) Prove that in a principal ideal domain, an element is an irreducible if and only if it is a prime. 6

(f) (i) Prove that the ring of integers Z is an Euclidean Domain. 2

(ii) Prove that every Euclidean Domain is a principal ideal domain. 4

4. Answer **any four** parts : 10×4=40

(a) (i) Let $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$. Prove that $Z(\sqrt{2})$ is a ring under the ordinary addition and multiplication of real numbers. 6

(ii) Let R be a ring. Prove that $a^2 - b^2 = (a+b)(a-b)$ for all a, b in R if and only if R is commutative. 4

(b) (i) Define a field. Prove that a finite integral domain is a field. Hence show that for any prime p, Z_p the ring of integers modulo p, is a field. 1+5+2=8

(ii) Show that 0 is the only nilpotent element in an integral domain. 2

(c) (i) Show that $S = \{a + ib \mid a, b \in Z, b \text{ is even}\}$ is a subring of $Z[i]$ but not an ideal of $Z[i]$. 4

(ii) Prove that the only ideals of a field F are $\{0\}$ and F itself. 2

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(iii) Show that $R[x]/\langle x^2+1\rangle$ is a field. 4

(d) (i) Let ϕ be a monomorphism from a ring R to a ring S.
Prove that kernel of ϕ is an ideal of R. 4

(ii) Let ϕ be a homomorphism from a ring R to a ring S. Prove that $R/Ker\,\phi \cong \phi(R)$. 6

(e) (i) If ϕ is an isomorphism from a ring R to a ring S, then show that ϕ^{-1} is an isomorphism from S to R. 5

(ii) Let R be a ring with unity e. Prove that the mapping $\phi : Z \to R$ given by $n \to ne$ is a ring homomorphism. 5

(f) Let F be a field and let $f(x)$ and $g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exist unique polynomials $q(x)$ and $r(x)$ in $F(x)$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $deg\, r(z) < deg\, g(x)$.

Hence show that if $a \in F$ and $f(x) \in F[x]$, then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$. 7+3=10

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