3 (Sem–1/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–1016

(Calculus)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** of the following questions : $1\times7=7$

   (a) Write down the n^{th} derivative of $\cos(5x+3)$.

   (b) Write when the graph of a function f is said to have vertical tangent at a point $P(c, f(c))$.

(c) Write down the value of $\underset{x \to +\infty}{Lt} x^n e^{-kx}$

(d) Evaluate $\int_0^{\pi/2} sin^6 x dx$

(e) In terms of marginal revenue and marginal cost, when is the profit maximized?

(f) For what purpose the disk and washer methods are used?

(g) Parameterize the curve $y = 4x^2$

(h) When the graph of a vector function $\vec{F}(t)$ is said to be smooth?

(i) Determine the values of t for which the vector function $\vec{F}(t) = \dfrac{\hat{i} + 2\hat{j}}{t^2 + 1}$ is continuous?

(j) State the geometrical significance of the scalar triple product of vectors $\vec{u}, \vec{v}$ and $\vec{w}$.

(k) Find $\int_0^{\pi} \left(t\hat{i} + 3\hat{j} - \sin t\,\hat{k}\right) dt$

(l) When a function f is said to be continuously differentiable on an interval I?

2. Answer **any four** of the following questions : 2×4=8

(a) Evaluate $\underset{x \to +\infty}{Lt} \dfrac{3x^3 - 5x + 9}{5x^3 + 2x^2 - 7}$

(b) Using Leibnitz's rule obtain the n^{th} derivative of $y = x^3 e^x$.

(c) By integration find the length of the circle $r = 2\sin\theta$.

(d) Let $\vec{F}(t) = \hat{i} + e^t\hat{j} + t^2\hat{k}$ and

$\vec{G}(t) = 3t^2\hat{i} + e^{-t}\hat{j} + 2t\hat{k}$,

then find $\frac{d}{dt}\{\vec{F}(t)\cdot\vec{G}(t)\}$.

(e) Find the tangent vector to the graph of the vector function $\vec{F}(t) = t^2\hat{i} + 2t\hat{j} + e^t\hat{k}$ at the point $t = -1$.

(f) State Kepler's laws of motion.

(g) Find the volume generated by revolving about OX, the area bounded by $y = x^3$ between $x = 0$ and $x = 2$.

(h) Find the length of the polar curve $r = e^{3\theta}, 0 \le \theta \le \pi/2$.

3. Answer ***any three*** of the following : 5×3=15

(a) Find the constants a and b that guarantee that the graph of the function defined by $f(x) = \frac{ax+5}{3-bx}$.

will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 3$.

(b) Evaluate : 2+3=5

(i) $\underset{x \to \pi/2^-}{Lt} (x - \pi/2) \tan x$

(ii) $\underset{x \to +\infty}{Lt} \left(1 + \frac{1}{2x}\right)^{3x}$

(c) If $I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, then prove that

$n(I_{n+1} + I_{n-1}) = 1$.

Hence evaluate $\int_0^{\pi/4} \tan^3 \theta \, d\theta$. 3+2=5

(d) A firm determines that x units of its product can be sold daily at p rupees per unit where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$.

Find the revenue function $R(x)$.

Find the profit function $P(x)$.

Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit. 1+1+3=5

(e) Show that a cone of radius r and height h has lateral surface area

$S = \pi r \sqrt{r^2 + h^2}$.

(f) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ in space, prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c}\,.\,\vec{a})\,\vec{b} - (\vec{b}\,.\,\vec{a})\,\vec{c}$.

(g) Use cylindrical shell method to find the volume of the solid generated when the region R under $y^2 = x$ and x-axis over the interval [0, 4] is revolved about the line $y = -1$.

(h) If the non-zero vector function $\vec{F}(t)$ is differentiable and has constant length, then prove that $\vec{F}(t)$ is orthogonal to the derivative vector $\vec{F}'(t)$.

Verify this result for

$\vec{F}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3\hat{k}$. 3+2=5

4. Answer ***any three*** of the following : 10×3=30

(a) State Leibnitz's theorem. Use it to show that if $y = e^{m\cos^{-1}x}$, then

$$\left(1-x^2\right)y_{n+2}-(2n+1)\,xy_{n+1}-\left(n^2+m^2\right)y_n=0$$

Hence find $y_n(0)$. 2+5+3=10

(b) Find the vertical and horizontal asymptotes (if any) of the graph of the function

$$f(x)=\frac{x^2-x-2}{x-3}.$$

Find where the graph is rising, where it is falling, determine concavity, locate all critical points and points of inflection. Finally sketch the graph.

(c) Obtain the reduction formula for $\int \sin^n x \, dx$.

Hence evaluate

(i) $\int_0^{\pi/2} \sin^n x \, dx$

(ii) $\int_0^{\pi/2} \sin^7 x \, dx$

5+3+2=10

(d) A boy standing at the edge of a cliff throws a ball upward at an angle of 30° with an initial speed of 64*ft*/s. Suppose that when the ball leaves the boy's hand, it is 48*ft* above the ground at the base of the cliff.

(i) What are the time of flight of the ball and its range?

(ii) What are the velocity of the ball and its speed at impact?

(iii) What is the highest point reached by the ball during its flight?

3+3+4=10

(e) (i) Find the area of the surface generated by revolving about the x-axis the top half of the cardioid $r = 1 + \cos\theta$. 5

(ii) Using disk method find the volume generated when the region bounded by the line $y = 4 - x$ and the x-axis on the interval $0 \le x \le 4$ revolve about the line $x = -2$. 5

(f) (i) Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$, given the acceleration $\vec{A}(t)$ and initial position and velocity vectors $\vec{R}(0)$ and $\vec{V}(0)$ as

$$\vec{A}(t) = t^2\hat{i} - 2\sqrt{t}\,\hat{j} + e^{3t}\hat{k}$$

$$\vec{R}(0) = 2\hat{i} + \hat{j} - \hat{k},\ \vec{V}(0) = \hat{i} - \hat{j} - 2\hat{k}.$$

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(ii) A particle moves along the parametric curve $x = 2t$, $y = t$. Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$ in terms of $\hat{U}_r$ and $\hat{U}_\theta$. 5

(g) (i) It is projected that t years from now, the population of a certain country will be $P(t) = 50e^{0\cdot02t}$ million.

At what rate will the population be changing with respect to time 10 years from now?

At what percentage of rate, will the population be changing with respect to time t years from now?

3+3=6

(ii) Find the length of the curve defined by $9x^2 = 4y^3$ between the points $(0, 0)$ and $(2\sqrt{3}, 3)$. 4

(h) A object moving along a smooth curve has velocity $\vec{V}$ given by $\vec{V} = \frac{ds}{dt}\hat{T}$.

Deduce the expression for acceleration in the form $\vec{A} = \frac{d^2s}{dt^2}\hat{T} + k\left(\frac{ds}{dt}\right)^2 \hat{N}$ where s is the arc length along the trajectory and k is the curvature. For an object moving along a helix with position vector $\vec{R}(t) = (\cos t, \sin t, t)$ at any instant t, find the tangential and normal components of acceleration.

5+5=10
