3 (Sem-1/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper: MAT-HC-1026

(Algebra)

Full Marks: 80

Time: Three hours

The figures in the margin indicate full marks for the questions.

Answer any ten :

 $1 \times 10 = 10$

(a) Find the polar representation of z = -3i.

(b) State De Moivre's theorem.

Let $z_0 = r(\cos t^* + i \sin t^*)$ be a complex number with r > 0 and $t^* \in [0, 2\pi)$. Write down the formula for n distinct n^{th} roots of z_0 .

- (d) Identify the quantifier, set of context and property in the statement, "Every student in this classroom is at least 5ft tall."
- (e) Define implication. Give an example.
- (f) Prove by contradiction "There is no greatest integer".
- (g) Let A and B be two sets, write when $A \times B = \phi$. Justify your answer.
- (h) What is domain and range for the function $f(x) = \tan x$.
- (i) What are the options about the solutions of a system of linear equations?
- Determine h such that the matrix $\begin{bmatrix} 2 & 3 & h \\ 6 & 9 & 5 \end{bmatrix}$ is the augmented matrix of a consistent linear system.
 - (k) State True **or** False with justification: "Whenever a system has free variables the solution set is infinite."

(1) Write down the system of equations that is equivalent to the vector equation

$$x_1\begin{bmatrix} -2\\3 \end{bmatrix} + x_2\begin{bmatrix} 8\\5 \end{bmatrix} + x_3\begin{bmatrix} 1\\-6 \end{bmatrix} - \begin{bmatrix} 0\\0 \end{bmatrix}.$$

(m) Define Pivot positions in a matrix.

(n) Prove
$$\vec{U} + \vec{V} = \vec{V} + \vec{U}$$
 for any $\vec{U} \cdot \vec{V}$ in \mathbb{R}^n .

(o) Write the system of equation as a matrix equation

$$3x_1 + x_2 - 5x_3 = 9$$
$$x_2 + 4x_3 = 0$$

(p) Given,
$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$
 $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Compute $x^T A^T$ and $A^T x^T$.

- (q) A is an $n \times n$ matrix. Prove statement (i) \Rightarrow statement (ii).
 - (i) A is an invertible matrix
 - (ii) $\exists a \ n \times n \text{ matrix } C \text{ s.t. } CA = I$

- (r) A is an $n \times n$ matrix

 Fill in the blank:

 If two rows of A are interchanged to produce B, then det B =_____.
- 2. Answer any five:

2×5=10

(a) If $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Express $z_1 z_2$ in polar form.

(b) Write the 'converse' and 'contrapositive' of the following statement:
"For real numbers x and y, if xy is an irrational number then either x is irrational or y is irrational."

- (c) Why may we use the contrapositive of a statement to prove the statement instead of direct proof? Justify using truth table.
- (d) Produce counter examples to disapprove the following:
 - (i) For $x, y \in \mathbb{R}$, |a| > |b| if a > b
 - (ii) For any $x \in \mathbb{R}$, $x^2 \ge x$

- Express the empty set as a subset of $\mathbb R$ (e) in two different ways.
 - Express N as the union of an infinite (f)number of finite sets I_n indexed by $n \in \mathbb{N}$.
- Give an example of a relation that is not reflexive, not transitive but is symmetric.
 - State True or False with justification: (h) An example of a linear combination of vectors \vec{v}_1 and \vec{v}_2 is $\frac{1}{2}\vec{v}_1$.
 - Prove that the following vectors are (i) linearly dependent

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
 and $\vec{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

Evaluate the determinant by using row reduction to Echelon form

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

3. Answer any four:

Compute
$$z = (1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$$
.

- (b) Prove that the power set of a set with n elements has 2^n elements. Write down the power set of $S = \{a, b\}$.
- (c) Prove that the equivalence classes of an equivalence relation on a set X induces a partition of X.
- (d) Prove $(1+x)^n \ge 1+nx$ for $x \in \mathbb{R}$ such that x > -1 and for each $n \in \mathbb{N}$. Give the name of this inequality.
- (e) Balance the chemical equation using vector equation approach the following reaction between potassium permanganate (KMnO₄) and manganese sulfate (MnSO₄) in water produces manganese dioxide, potassium sulfate and sulfuric acid.

The unbalanced equation is

 $KMnO_4 + MnSO_4 + H_2O \rightarrow MnO_2 + K_2SO_4 + H_2SO_4$

(f) Find the value of h for which the set of vectors is linearly dependent

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix} \quad \begin{array}{c} -2 \\ 2 \\ \end{array} \quad \begin{array}{c} 7 \\ 2 \\ \end{array} \quad \begin{array}{c} 1 \\ 2 \\ \end{array} \quad \begin{array}{c} -6 + 7 \\ 2 \\ \end{array}$$

- (g) Let A be an $m \times n$ matrix. Prove that the following statements are logically equivalent.
 - (i) For each $b \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
 - (ii) Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.
 - (iii) The columns of A span \mathbb{R}^m .
 - (iv) A has a pivot position in every row.

Use Cramer's rule to compute the solutions to the system

$$2x_1 + x_2 = 7$$

$$-3x_1 + x_3 = -8$$

$$x_2 + 2x_3 = -3$$



4. Answer any four:

(a) (i) Prove
$$\prod_{\substack{1 \le k \le n-1 \\ \gcd(k, n)=1}} \sin \frac{k\pi}{n} = \frac{1}{2^{\phi(n)}}$$

whenever n is not a power of a prime.

(ii) Solve the equation

$$z^7 - 2iz^4 - iz^3 - 2 = 0 5$$

(b) For any three sets A, B and C, show that

(i)
$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

(ii)
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
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(c) Define graph of a function verify that the set $\{(x,y) \in \mathbb{R}^2 : x = |y|\}$ is not the graph of any function. Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx + c, a \neq 0$. Show that the function is neither one-one nor onto.

7/2+2

(d) Let
$$X = \mathbb{R}$$
 and let

 $R = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$. When $x \in \mathbb{R}$ is related to $y \in \mathbb{R}$? Define reflexive, symmetric, antisymmetric and transitive relation with examples.

2+2+2+2+2=10

(e) If $A \subseteq N$, what is the least element of A? State and prove Division Algorithm. 2+1+7=10

(f) (i) Solve the system:
$$x_1 - 3x_2 + 4x_3 = -4$$

$$3x_1 - 7x_2 + 7x_3 = -8$$

$$-4x_1 + 6x_2 - x_3 = 7$$

(ii) Suppose the system
$$x_1 + 3x_2 = f$$
$$cx_1 + dx_2 = g$$

is consistent for all possible values of f and g, what can you say about the co-efficients c and d. Justify.

(iii) Suppose a 3 × 5 co-efficient matrix for a system has three pivot columns. Is the system consistent?

Justify.

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Contd.

(g) (i) If
$$\vec{U} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
 $\vec{V} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Display \vec{U} , \vec{V} , $\vec{U} - \vec{V}$ using arrows on an xy graph.

(ii) List five vectors in the span $\{\bar{v}_1, \bar{v}_2\}$

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \qquad 2$$

(iii) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span \mathbb{R}^4 ? Justify.

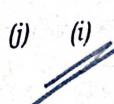
(h) (i) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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- (ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every possible \vec{b} if A is a 3×2 matrix with two pivot positions?
- Prove that if a set contains more vectors than the number of entries in each vector then the set is linearly dependent.
- (i) (i) Define linear transformation. Give an example.
 - (ii) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then prove T maps \mathbb{R}^n onto \mathbb{R}^m if and only if the columns of A span \mathbb{R}^m .
 - (iii) Find the standard matrix of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which is a horizontal shear transformation that leaves e_1 unchanged and maps e_2 into $e_2 + 3e_1$.
 - (iv) Show that T is a linear transformation by finding a matrix that implements the mapping

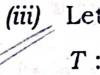
$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$



Find the inverse of the matrix A (if it exists) by performing suitable row operations on the augmented matrix [A:I] where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$

(ii) Find the volume of the parallelopiped with one vertex at the origin and adjacent vertices at (1, 0, -2), (1, 2, 4) and (7, 1, 0).



(iii) Let the transformation

 $T: \mathbb{R}^2 \to \mathbb{R}^2$ be determined by a 2×2 matrix A. Prove that if S is a parallelogram in \mathbb{R}^2 then $\{\text{area of } T(S)\} = /\det A / \{\text{area of } S\}$

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