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3 (Sem-2/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper: MAT-HC-2016

(Real Analysis)

Full Marks: 80

Time: Three hours

The figures in the margin indicate full marks for the questions.

1. Answer any ten questions: 1×10=10

(a) Find the infimum of the set

$$\left\{1-\frac{(-1)^n}{n}:n\in N\right\}$$

- (b) If A and B are two bounded subsets of R, then which one of the following is true?
 - (i) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$
 - (ii) $sup(A \cup B) = sup A + sup B$

Contd.

- (iii) $\sup(A \cup B) = \sup A \cdot \sup B$ (iv) $\sup(A \cup B) = \sup A \cup \sup B$
- (c) There does not exist a rational number x such that $x^2 = 2$. (Write True or False)
- (d) The set Q of rational numbers is uncountable. (Write True or False)
- (e) If $I_n = \left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = 2$
- (f) The convergence of {|x_n|} imply the convergence of {x_n}.

(Write True or False)

- (g) What are the limit points of the sequence $\{x_n\}$, where $x_n = 2 + (-1)^n$, $n \in \mathbb{N}$?
- (h) If $\{x_n\}$ is an unbounded sequence, then there exists a properly divergent subsequence. (Write True or False)
- A convergent sequence of real numbers is a Cauchy sequence.

(Write True or False)

- (i) If $0 < \alpha < 1$ then $\lim_{n \to \infty} a^n = ?$
- (k) The positive term series $\sum \frac{1}{n^p}$ is convergent if and only if
 - (i) p > 0
 - (ii) p>1
 - (iii) 0 < p < 1
 - (iv) $p \le 1$

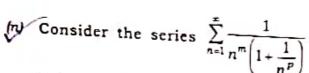
(Write correct one)

- Define conditionally convergent of a series.
- (m) If $\{x_n\}$ is a convergent monotone sequence and the series $\sum_{n=1}^{\infty} y_n$ is convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ is also convergent.

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where m and p are real numbers under which of the following conditions does the above series convergent?

- (i) m > 1
- (ii) 0 < m < 1 and p > 1
- (iii) $0 \le m \le 1$ and $0 \le p \le 1$
- (iv) m=1 and p>1
- (o) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers defined by $x_1 = 1$, $y_1 = \frac{1}{2}$, $x_{n+1} = \frac{x_n + y_n}{2}$ and $y_{n+1} = \sqrt{x_n y_n} \ \forall n \in \mathbb{N}$ then which one of the following is true?
 - (i) $\{x_n\}$ is convergent, but $\{y_n\}$ is not convergent
 - (ii) $\{x_n\}$ is not convergent, but $\{y_n\}$ is convergent

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- (iii) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n$
- (iv) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$
- 2. Answer any five parts:
 - (a) If a and b are real numbers and if a < b, then show that $a < \frac{1}{2}(a+b) < b$.
 - (b) Show that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is bounded.
 - (c) If $\{x_n\}$ converges in \mathbb{R} , then show that $\lim_{n\to\infty} x_n = 0$
 - (d) Show that the series 1+2+3+...., is not convergent.
 - (e) Test the convergence of the series:

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

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Contd.

 $2 \times 5 = 10$

- (f) State Cauchy's integral test of convergence.
- (g) State the completeness property of \mathbb{R} and find the $\sup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.
- (h) Does the Nested Interval theorem hold for open intervals? Justify with a counter example.
- 3. Answer any four parts: 5×4=20
 - (a) If x and y are real numbers with x < y, then prove that there exists a rational number r such that x < r < y.</p>
 - (b) Show that a convergent sequence of real numbers is bounded.
 - (c) Prove that $\lim_{n\to\infty} \left(n^{\frac{1}{n}}\right) = 1$.
 - (d) $\{x_n\}$ be a sequence of real numbers that converges to x and suppose that $x_n \ge 0$. Show that the sequence $\{\sqrt{x_n}\}$ of positive square roots converges and $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

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- (e) Show that every absolutely convergent series is convergent. Is the converse true? Justify. 4+1=5
- Using comparison test, show that the series $\sum (\sqrt{n^4+1} \sqrt{n^4-1})$ is convergent.
- (g) State Cauchy's root test. Using it, test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$$

1+4=5

(h) Show that the sequence defined by the recursion formula

$$u_{n+1} = \sqrt{3u_n}, \ u_1 = 1$$

is monotonically increasing and bounded. Is the sequence convergent? 2+2+1=5



Answer any four parts :

10×4=40

- (a) Show that the sequence $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}$ is convergent and $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$ which lies between 2 and 3.
- (b) (i) Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n$.

Show that $\{y_n\}$ is convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$ 5

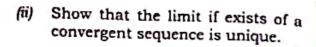
- (ii) What is an alternating series? State Leibnitz's test for alternating series. Prove that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....\infty$ is a conditionally convergent series. 1+1+3=5
- (c) Test the convergence of the series $1+a+a^2+.....+a^n+.....$

(d) (i) Using Cauchy's condensation test, discuss the convergence of the

series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ 5

- (ii) Define Cauchy sequence of real numbers. Show that the sequence $\left\{\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right\} \quad \text{is} \quad \text{a}$ Cauchy sequence.
- (e) (i) Show that a convergent sequence of real numbers is a Cauchy sequence.
 - (ii) Using Cauchy's general principle of convergence, show that the sequence $\left\{1+\frac{1}{2}+.....+\frac{1}{n}\right\}$ is not convergent.
 - (f) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound.

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- State and prove p-series. *(g)*
- Test the convergence of the series

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \dots + \frac{n^2 - 1}{n^2 + 1}x^n + \dots + (x > 0)$$

5

- (ii) If $\{x_n\}$ is a bounded increasing sequence then show that $\lim_{n\to\infty}x_n=\sup\{x_n\}$ 5
- (i) Show that a bounded sequence of real numbers has a convergent subsequence.
 - State and prove Nested Interval theorem.
- (i) Show that Cauchy sequence of real numbers is bounded.

Test the convergence of the series

$$x^{2} + \frac{2^{2}}{3.4}x^{4} + \frac{2^{2}.4^{2}}{3.4.5.6}x^{6} + \frac{2^{2}.4^{2}.6^{2}}{3.4.5.6.7.8}x^{6} + \dots (x > 0)$$

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