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3 (Sem-4/CBCS) MAT HC 3

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any ten** : $1 \times 10 = 10$

(a) The set Z of integers under ordinary addition and multiplication is a commutative ring with unity 1. What are the units of Z ?

(b) What is the trivial subring of R ?

Contd.

(c) What are the elements of $\mathbb{Z}_3[i]$?

(d) Give the definition of zero divisor.

(e) Give an example of a commutative ring without zero divisors that is not an integral domain.

(f) What is the characteristic of an integral domain?

(g) Why is the ideal $\langle x^2 + 1 \rangle$ not prime in $\mathbb{Z}_2[x]$?

(h) Find all maximal ideals in \mathbb{Z}_8 .

(i) Is the mapping from \mathbb{Z}_5 to \mathbb{Z}_{30} given by $x \rightarrow 6x$ is a ring homomorphism?

(j) If ϕ is an isomorphism from a ring R onto a ring S , then ϕ^{-1} is an isomorphism from S onto R .
Write True or False.

(k) Is the ring $2z$ isomorphic to the ring $3z$?

(l) Let $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ is $z_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

(m) Why is the polynomial

$$3x^5 + 15x^4 - 20x^3 + 10x + 20$$

irreducible over \mathbb{Q} ?

(n) Give the definition of Euclidean domain.

(o) State the second isomorphism theorem for rings.

2. Answer any five :

2×5=10

- (a) Define ring. What is the unity of a polynomial ring $Z[x]$?
- (b) Prove that in a ring R , $(-a)(-b) = ab$ for all $a, b \in R$.
- (c) Prove that set S of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with a and b , forms a sub-ring of the ring R of all 2×2 matrices having elements as integers.
- (d) Let R be a ring with unity 1 . If 1 has infinite order under addition, then the characteristic of R is 0 . If 1 has order n under addition, then prove that the characteristic of R is n .
- (e) Let $z/4z = \{0 + 4z, 1 + 4z, 2 + 4z, 3 + 4z\}$.
Find $(2 + 4z) + (3 + 4z)$ and $(2 + 4z)(3 + 4z)$.

(f) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$ and let ϕ be

the mapping defined as $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \rightarrow a - b$.

Show that ϕ is a homomorphism.

(g) Let $f(x) = 4x^3 + 2x^2 + x + 3$ and

$$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$$

where $f(x), g(x) \in \mathbb{Z}_5[x]$.

Compute $f(x) + g(x)$ and $f(x) \cdot g(x)$.

(h) Prove that in an integral domain, every prime is an irreducible.

3. Answer **any four** :

5×4=20

(a) Define a sub-ring. Prove that a non-empty subset S of a ring R is a sub-ring if S is closed under subtraction and multiplication, that is if $a - b$ and ab are in S whenever a and b are in S .

1+4=5

(b) Prove that the ring of Gaussian integers $Z[i] = [a + ib \mid a, b \in Z]$ is an integral domain.

(c) Let R be a commutative ring with unity and let A be an ideal of R . Then prove that R/A is an integral domain if and only if A is prime.

(d) If D is an integral domain, then prove that $D[x]$ is an integral domain.

(e) (i) If R is commutative ring then prove that $\phi(R)$ is commutative, where ϕ is an isomorphism on R . 3

(ii) If the ring R has a unity 1 , $S \neq \{0\}$ and $\phi : R \rightarrow S$ is onto, then prove that $\phi(1)$ is the unity of S . 2

(f) Let $f(x) \in Z[x]$. If $f(x)$ is reducible over Q , then prove that it is reducible over Z .

(g) Consider the ring

$$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}. \text{ Show that}$$

$\phi : \mathbb{C} \rightarrow S$ is given by

$$\phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ is a ring}$$

isomorphism.

(h) Prove that $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, the ring of Gaussian integers is an Euclidean domain.

4. Answer **any four** :

10×4=40

(a) (i)

Prove that the set of all continuous real-valued functions of a real variable whose graphs pass through the point (1, 0) is a commutative ring without unity under the operation of pointwise addition and multiplication [that is, the operations $(f + g)(a) = f(a) + g(a)$ and $(f \cdot g)(a) = f(a) \cdot g(a)$].

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(ii) Prove that if a ring has a unity, it is unique and if a ring element has an inverse, it is unique. 4

(b) Define a field. Is the set I of all integers a field with respect to ordinary addition and multiplication? Let

$Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q\}$. Prove that

$Q[\sqrt{2}]$ is a field. 2+1+7=10

(c) (i) Prove that the intersection of any collection of subrings of a ring R is a sub-ring of R . 5

(ii) Let R be a commutative ring with unity and let A be an ideal of R . Prove that R/A is a field if A is maximal. 5

(d) Define factor ring. Let R be a ring and let A be a subring of R . Prove that the set of co-sets $\{r + A \mid r \in R\}$ is a ring under the operation

$$(s + A) + (t + A) = (s + t) + A \text{ and}$$

$$(s + A)(t + A) = st + A \text{ if and only if } A \text{ is an ideal of } R. \quad 1+5+4=10$$

(e) (i) Let ϕ be a ring homomorphism from R to S . Prove that the mapping from $R/\ker \phi$ to $\phi(R)$, given by $r + \ker \phi \rightarrow \phi(r)$ is an isomorphism. 5

(ii) Let R be a ring with unity and the characteristic of R is $n > 0$. Prove that R contains a subring isomorphic to Z_n . If the characteristic of R is 0, then prove that R contains a sub-ring isomorphic to Z . 3+2=5

(f) Let F be a field and let $p(x) \in F[x]$. Prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F .

(g) Let F be a field and let $f(x)$ and $g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exists unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $\text{degr}(x) < \text{deg } g(x)$. With the help of an example verify the division algorithm for $F[x]$. 7+3=10

(h) (i) If F is a field, then prove that $F[x]$ is a principal ideal domain. 5

(ii) Let F be a field and let $p(x), a(x), b(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) | a(x)b(x)$, then prove that $p(x) | a(x)$ or $p(x) | b(x)$. 5

(i) Prove that every principal ideal domain is a unique factorization domain.

(j) (i) Prove that every Euclidean domain is a principal ideal domain. 5

(ii) Show that the ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$$

is an integral domain but not a unique factorization domain. 5