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3 (Sem–5/CBCS) MAT HC 1 (N/O)

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–5016

(For New Syllabus)

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** questions from the following : $1\times7=7$

(a) Describe the domain of definition of the function $f(z)=\dfrac{z}{z+\overline{z}}$.

(b) What is the multiplicative inverse of a non-zero complex number $z=(x, y)$?

Contd.

(c) Verify that $(3, 1)\ (3, -1)\ \left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

(d) Determine the accumulation points of the set $Z_n = \frac{i}{n}\ (n = 1, 2, 3, \ldots)$.

(e) Write the Cauchy-Riemann equations for a function $f(z) = u + iv$.

(f) When a function f is said to be analytic at a point?

(g) Determine the singular points of the function $f(z) = \frac{2z+1}{z(z^2+1)}$.

(h) $exp(2 \pm 3\pi i)$ is

(i) $-e^2$

(ii) e^2

(iii) $2e$

(iv) $-2e$ *(Choose the correct answer)*

(i) The value of $\log(-1)$ is

(i) 0

(ii) $2n\pi i$

(iii) πi

(iv) $-\pi i$ (Choose the correct answer)

(j) If $z = x + iy$, then $\sin z$ is

(i) $\sin x \cosh y + i \cos x \sinh y$

(ii) $\cos x \cosh y - i \sin x \sinh y$

(iii) $\cos x \sinh y + i \sin x \cosh y$

(iv) $\sin x \sinh y - i \cos x \cosh y$

(Choose the correct answer)

(k) If $\cos z = 0$, then

(i) $z = n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(ii) $z = \frac{\pi}{2} + n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(iii) $z = 2n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(iv) $z = \frac{\pi}{2} + 2n\pi, (n = 0, \pm 1, \pm 2, \ldots)$

(Choose the correct answer)

(l) If z_0 is a point in the z-plane, then $\lim_{z \to \infty} f(z) = \infty$ if

(i) $\lim_{z \to 0} \dfrac{1}{f(z)} = \infty$

(ii) $\lim_{z \to 0} f\left(\dfrac{1}{z}\right) = 0$

(iii) $\lim_{z \to 0} \dfrac{1}{f(z)} = 0$

(iv) $\lim_{z \to 0} \dfrac{1}{f\left(\dfrac{1}{z}\right)} = 0$

(Choose the correct answer)

2. Answer ***any four*** questions from the following : 2×4=8

(a) Reduce the quantity $\dfrac{5i}{(1-i)(2-i)(3-i)}$ to a real number.

(b) Define a connected set and give *one* example.

(c) Find all values of z such that $exp(2z-1)=1$.

(d) Show that $log(i^3) \neq 3 log\, i$.

(e) Show that

$$2 sin(z_1+z_2)\, sin(z_1-z_2) = cos 2z_2 - cos 2z_1.$$

(f) If z_0 and w_0 are points in the z plane and w plane respectively, then prove that $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

(g) State the Cauchy integral formula. Find $\frac{1}{2\pi i} \int_C \frac{1}{z-z_0} dz$ if z_0 is any point interior to simple closed contour C.

(h) Show that $\int_0^{\frac{\pi}{6}} e^{i2t}\, dt = \frac{\sqrt{3}}{4} + \frac{i}{4}$.

3. Answer *any three* questions from the following : $5 \times 3 = 15$

(a) (i) If a and b are complex constants, use definition of limit to show that

$$\lim_{z \to z_0} (az + b) = az_0 + b.$$ 2

(ii) Show that

$$\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2 \text{ does not exist.}$$ 3

(b) Suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$.

Prove that $\lim_{z \to z_0} [f(z) F(z)] = w_0 W_0$.

(c) (i) Show that for the function $f(z) = \bar{z}$, $f'(z)$ does not exist anywhere. 3

(ii) Show that $\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$. 2

(d) (i) Show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere. 3

(ii) Find all roots of the equation $\log z = i\frac{\pi}{2}$. 2

(e) If a function f is analytic at all points interior to and on a simple closed contour C, then prove that $\int_C f(z)dz = 0$.

(f) Evaluate : 2½+2½=5

(i) $\int_C \frac{e^{-z}}{z-(\pi i/2)} dz$

(ii) $\int_C \frac{z}{2z+1} dz$

where C denotes the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(g) Prove that any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \ (a_n \neq 0)$$

of degree $n (n \geq 1)$ has at least one zero.

(h) Find the Laurent series that represents the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ in the domain $0 < |z| < \infty$.

4. Answer ***any three*** questions from the following : 10×3=30

(a) (i) If a function f is continuous throughout a region R that is both closed and bounded, then prove that there exists a non-negative real number μ such that $|f(z)| \leq \mu$ for all points z in R, where equality holds for at least one such z. 4

(ii) Let a function $f(z) = u(x, y) + iv(x, y)$ be analytic throughout a given domain D. If $|f(z)|$ is constant throughout D, then prove that $f(z)$ must be constant there too. 3

(iii) Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is entire. 3

(b) (i) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$ 3

(ii) Show that $f'(z)$ does not exist at any point if $f(z) = 2x + ixy^2$. 3

(iii) If a function f is analytic at a given point, then prove that its derivatives of all orders are analytic there too. 4

(c) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε-neighbourhood of a point $z_0 = x_0 + iy_0$. If u_x, u_y, v_x, v_y exist everywhere in the neighbourhood, and these partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0), then prove that $f'(z_0)$ exist and $f'(z_0) = u_x + iv_x$ where the right hand side is to be evaluated at (x_0, y_0).

Use it to show that for the function $f(z) = e^{-x} \cdot e^{-y}$, $f''(z)$ exists everywhere and $f''(z) = f(z)$. 6+4=10

(d) (i) Prove that the existence of the derivative of a function at a point implies the continuity of the function at that point.

With the help of an example show that the continuity of a function at a point does not imply the existence of derivative there. 3+5=8

(ii) Find $f'(z)$ if

$$f(z)=\frac{z-1}{2z+1}\left(z\neq-\frac{1}{2}\right).$$ 2

(e) (i) Prove that $\int_C \frac{dz}{z}=\pi i$ where C is the right-hand half $z=2e^{i\theta}$ $\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$ of the circle $|z|=2$ from $z=-2i$ to $z=2i$. 5

(ii) If a function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense, then prove that

$$f'(z)=\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds$$ where s denotes points on C and z is interior to C. 5

(f) (i) Evaluate $I = \int_C z^{a-1}\, dz$

where C is the positively oriented circle $z = Re^{i\theta} \ (-\pi \le \theta \le \pi)$ about the origin and a denote any non-zero real number.

If a is a non-zero integer n, then what is the value of $\int_C z^{n-1}\, dz$?

4+1=5

(ii) Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If μ is a non-negative constant such that $|f(z)| \le \mu$ for all point z on C at which $f(z)$ is defined, then prove that $\left| \int_C f(z)\, dz \right| \le \mu L$.

Use it to show that $\left| \int_C \frac{dz}{z^2 - 1} \right| \le \frac{\pi}{3}$ where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the Ist quadrant.

3+2=5

(g) (i) Apply the Cauchy-Goursat theorem to show that $\int_C f(z) = 0$ when the contour C is the unit circle $|z| = 1$, in either direction and $f(z) = ze^{-z}$. 4

(ii) If C is the positively oriented unit circle $|z| = 1$ and $f(z) = exp(2z)$ find $\int_C \frac{f(z)}{z^4} dz$. 3

(iii) Let z_0 be any point interior to a positively oriented simple closed curve C. Show that

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = 0, \ (n = 1, 2, \ldots).$$ 3

(h) (i) Suppose that $z_n = x_n + iy_n$, $(n = 1, 2, \ldots)$ and $z = x + iy$. Prove that $\lim_{n \to \infty} z_n = z$ if and only if

$$\lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y.$$ 5

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \ (|z| < \infty)$$

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