Total number of printed pages–32

3 (Sem–5/CBCS) MAT HE 1/HE 2/HE 3

2022

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

DSE (H)–1

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

PART–A

1. Choose the correct option in each of the following questions : ***(any ten)*** $1\times10=10$

   (i) Number of integers which are less than and co-prime to 108 is

   (a) 18

   (b) 17

Contd.

(c) 15

(d) 36

(ii) The number of positive divisors of a perfect square number is

(a) odd

(b) even

(c) prime

(d) Can't say

(iii) If $100! \equiv x(mod 101)$, then x is

(a) 99

(b) 100

(c) 101

(d) None of the above

(iv) The solution of pair of linear congruences $2x \equiv 1 (mod\ 5)$ and $x \equiv 4\ (mod\ 3)$ is

(a) $x \equiv 13(mod 5)$

(b) $x \equiv 28(mod 5)$

(c) $x \equiv 13(mod 15)$

(d) $x \equiv 13(mod 3)$

(v) If $a = qb$ for some integer q and $a, b \neq 0$, then

(a) b divides a

(b) a divides b

(c) $a = b$

(d) None of the above

(vi) If a and b are any two integers, then there exists some integres x and y such that

(a) $gcd(a,b) = ax + by$

(b) $gcd(a,b) = ax - by$

(c) $gcd(a,b) = ax^n + by^m$

(d) $gcd(a,b) = (ax + by)^n$

(vii) The linear diophantine equation $ax + by = c$ with $d = gcd(a,b)$ has a solution in integers if and only if

(a) $d | c$

(b) $c | d$

(c) $d | (ax+by)$

(d) Both (a) and (c)

(viii) If a positive integer n divides the difference of two integers a and b, then

(a) $a \equiv b(\mathrm{mod}\, n)$

(b) $a = b(\mathrm{mod}\, n)$

(c) $a \equiv n(\mathrm{mod}\, b)$

(d) None of the above

(ix) The set of integers such that every integer is congruent modulo m to exactly one integer of the set is called ____ modulo m. *(Fill in the blank)*

(a) Reduced residue system

(b) Complete residue system

(c) Elementary residue system

(d) None of the above

(x) Which of the following statement is false ?

(a) There is no pattern in prime numbers

(b) No formulae for finding prime numbers

(c) Both (a) and (b)

(d) None of the above

$\phi(P) = P - P^{k-1}$

(xi) The reduced residue system is ____ of complete residue system.

(a) compliment

(b) subset

(c) not a subset

(d) Both (a) and (c)

(xii) The unit place digit of 137^{93} is

(a) 7

(b) 9

(c) 3

(d) 1

(xiii) Euler phi-function of a prime number p is

(a) p

(b) $p-1$

(c) $p/2-1$

(d) None of the above

Contd.

(xiv) Which theorem states that "if p is prime, then $(p-1)! \equiv -1 (mod\, p)$" ?

(a) Dirichlet's theorem

(b) Wilson's theorem

(c) Euler's theorem

(d) Fermat's little theorem

(xv) Let p be an odd prime. Then $x^2 \equiv -1 (mod\, p)$ has a solution if p is of the form

(a) $4k+1$

(b) $4k$

(c) $4k+3$

(d) None of the above

(xvi) Let m be a positive integer. Two integers a and b are congruent modulo m if and only if

(a) $m \mid (a-b)$

(b) $m \mid (a+b)$

(c) $m \mid (ab)$

(d) Both *(b)* and *(c)*

(xvii) If $ac \equiv bc (mod\, m)$ and $d = gcd(m, c)$

(a) $a \equiv b\left(mod \dfrac{m}{d}\right)$

(b) $a \equiv c\left(mod \dfrac{m}{d}\right)$

(c) $a \equiv m (mod\, b)$

(d) $a \equiv m\left(mod \dfrac{b}{a}\right)$

(xviii) If a is a whole number and p is a prime number, then according to Fermat's theorem

(a) $a^p - a$ is divisible by p

(b) $a^p - 1$ is divisible by p

(c) $a^{p-1} - 1$ is divisible by p

(d) $a^{p-1} - a$ is divisible by p

$a^{\phi(n)} \equiv 1 (mod\, n)!$
$\phi(100) = 40$
$(3^{10}) \equiv 1 (mod\, p$

2. Answer ***any five*** questions : 2×5=10

(a) Find last two digits of 3^{100} in its decimal expansion.

(b) If p and q are positive integers such that $gcd(p,q)=1$, then show that $gcd(a+b,\ a-b)=1$ *or* 2.

(c) Find the solution of the following linear Diophantine equation $8x-10y=42$.

(d) If p and q are any two real numbers, then prove that $[p]+[q]\leq[p+q]$ (where $[x]$ denotes the greatest integer less *or* equal to x).

(e) If m and n are integers such that $(m,n)=1$, then $\varphi(mn)=\varphi(m)\varphi(n)$.

(f) Find (7056).

(g) If $a\equiv b(mod\,n)$ and $m\mid n$, then show that $a\equiv b(mod\,m)$.

(h) List all primitive roots modulo 7.

(i) If $n=p_1^{k_1}p_2^{k_2}\ldots.p_m^{k_m}$ is the prime factorization of $n>1$, then prove that $\tau(n)=(k_1+1)(k_2+1)\ldots\ldots\ldots(k_n+1)$.

(j) Evaluate the exponent of 7 in 1000 !

3. Answer **any four** questions : 5×4=20

(a) If p is a prime, then prove that $\varphi(p!)=(p-1)\varphi((p-1)!)$

(b) Show that, the set of integers $\{1,5,7,11\}$ is a reduced residue system (RRS) modulo 12.

(c) Solve the following simultaneous congruence :

$$x\equiv 2(mod\,3)$$
$$x\equiv 2(mod\,2)$$
$$x\equiv 3(mod\,5)$$

(d) For $n=p^k, p$ is a prime, prove that $n=\sum_{d|n}\varphi(d)$, where $\sum_{d|n}$ denotes the sum over all positive divisors of n.

(e) If p_n is the n^{th} prime, then show that $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_n}$ is not an integer.

(f) Let n be any integer > 2. Then $\varphi(n)$ is even.

(g) Show that if $a_1,a_2,\ldots..,a_{\varphi(m)}$ is a RRS modulo m, where m is a positive integer with $m\neq 2$, then $a_1+a_2+\ldots+a_{\varphi(m)}\equiv 0(mod\,m)$.

(h) Show that $10! + 1$ is divisible by 11.

PART–B

Answer **any four** of the following questions: $10\times4=40$

4. (a) If $a, b \neq 0$ and c be any three integers and $d = gcd(a,b)$. Then show that $ax + by = c$ has a solution iff $d \mid c$.

Furthermore, show that if x_0 and y_0 is a particular solution of $ax + by = c$, then any other solution of the equation is $x' = x_0 - \frac{b}{d}t$ and $y' = y_0 + \frac{a}{d}t$, t is an integer. 7

(b) Find the general solution of $10x - 8y = 42$; $x, y \in Z$ 3

5. (a) Show that an odd prime p can be represented as sum of two squares iff $p \equiv 1(mod\,4)$. 7

(b) Find all positive solutions of $x^2 + y^2 = z^2$, where $0 < z < 30$. 3

6. (a) Prove that the quadratic congruence $x^2 + 1 \equiv 0(mod\,p)$, where p is an odd prime, has a solution iff $p \equiv 1(mod\,4)$. 5

(b) If p is prime and a is an integer not divisible by p, prove that $a^{p-1} \equiv 1\,(mod\,p)$. 5

7. State and prove Chinese remainder theorem. Also find all integers that leave a remainder of 4 when divided by 11 and leaves a remainder of 3 when divided by 17.

8. (a) For each positive integer $n \geq 1$, show that $\sum_{d|n} \mu(d) = \begin{cases} 1, \text{ if } n = 1 \\ 0, \text{ if } n > 1 \end{cases}$ 5

(b) If k denotes the number of distinct prime factors of positive integer n. Prove that $\sum_{d|n} |\mu(d)| = 2^k$ 5

9. (a) If p is a prime, prove that $\varphi(p^k) = p^k - p^{k-1}$, for any positive integer k. For $n > 2$, show that $\varphi(n)$ is an even integer. 3+2=5

(b) State Mobius inversion formula. If the integer $n>1$ has the prime factorization. If $n=p_1^{k_1}p_2^{k_2}....p_s^{k_s}$, then prove that

$\sum_{d|n}\mu(d)\sigma(d)=(-1)^s p_1 p_2 ... p_s$. 5

10. If x be any real number. Then show that 1+3+3+3=10

(a) $[x]\leq x<[x]+1$

(b) $[x+m]=[x]+m$, m is any integer

(c) $[x]+[-x]=\begin{cases}0, & \text{if } x \text{ is an integer}\\ -1, & \text{otherwise}\end{cases}$

(d) $\left[\frac{[x]}{m}\right]=\left[\frac{x}{m}\right]$, if m is a positive integer

11. (a) If $a_1, a_2, ..., a_m$ is a complete residue system modulo m, and if k is a positive integer with $(k,m)=1$ then $ka_1+b, ka_2+b, ..., ka_m+b$, is a complete residue system modulo m for any integer b. 5

(b) Examine whether the following set forms a complete residue system or a reduced residue system :

$\{-3,14,3,12,37,56,-1\}(mod 7)$ 5

12. (a) If $n\geq 1$ is an integer then show that

$$\Pi_{d|n} d = n^{\frac{\tau(n)}{2}}$$ 3

(b) If f and g are two arithmetic functions, then show that the following conditions are equivalent : 7

(i) $f(n)=\sum_{d|n} g(d)$

(ii) $g(n)=\sum_{d|n}\mu(d)f\left(\frac{n}{d}\right)=\sum_{d|n}\mu\left(\frac{n}{d}\right)f(d)$

13. (a) If n is a positive integer with $n\geq 2$, such that $(n-1)!+1\equiv 0 \pmod n$, then show that n is prime. 5

(b) Show that if p is an odd prime, then $2(p-3)!\equiv -1 \pmod p$. 5

Contd.