Total number of printed pages–16

3 (Sem–5/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

OPTION–A

(For New Syllabus)

Paper : MAT–HC–5016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

(a) Which point on the Riemann sphere represents ∞ of the extended complex plane $\mathbb{C}\cup\{\infty\}$?

(b) A set $S\subseteq\mathbb{C}$ is closed if and only if S contains each of its ________ points. *(Fill in the gap)*

Contd.

(c) Write down the polar form of the Cauchy-Riemann equations.

(d) The function $f(z) = \sinh z$ is a periodic function with a period ______ .

(Fill in the gap)

(e) Define a simple closed curve.

(f) Write down the value of the integral $\int_C f(z)\,dz$, where $f(z) = ze^{-2}$ and C is the circle $|z| = 1$.

(g) Find $\lim_{n\to\infty} z_n$, where $z_n = -1 + i\frac{(-1)^n}{n^2}$.

2. Answer the following questions : 2×4=8

(a) Let $f(z) = i\frac{z}{2}$, $|z| < 1$. Show that $\lim_{z\to 1} f(z) = \frac{i}{2}$, using $\varepsilon - \delta$ definition.

(b) Show that all the zeros of $\sinh z$ in the complex plane lie on the imaginary axis.

(c) Evaluate the contour integral

$$\int_C \frac{dz}{z},$$ where C is the semi circle $z = e^{i\theta}, \quad 0 \le \theta \le \pi$

(d) Using Cauchy's integral formula, evaluate

$$\int_C \frac{e^{2z}}{z^4} dz,$$ where C is the circle $|z| = 1$.

3. Answer ***any three*** questions from the following : 5×3=15

(a) Find all the fourth roots of -16 and show that they lie at the vertices of a square inscribed in a circle centered at the origin.

(b) Suppose $f(z) = u(x, y) + iv(x, y)$, $(z = x + iy)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then prove the following :

$$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0,$$

$$\lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0,$$ if and only if $\lim_{z \to z_0} f(z) = w_0$.

(c) (i) Show that the function $f(z) = Re\,z$ is nowhere differentiable.

(ii) Let $T(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$.

Show that $\lim_{z \to \infty} T(z) = \infty$ if $c = 0$.

3+2=5

(d) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3 - 1} dz \right| \leq \frac{6\pi}{7}$$

(e) State and prove fundamental theorem of algebra.

4. Answer ***any three*** questions from the following : 10×3=30

(a) (i) Show that $exp(z + \pi i) = -exp(z)$

1

(ii) Show that

$log(-1+i)^2 \neq 2log(-1+i)$ 2

(iii) Show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$ 2

(iv) Show that a set $S \subseteq \mathbb{C}$ is unbounded if and only if every neighbourhood of the point at infinity contains at least one point of S. 5

(b) (i) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$. Using the definition of derivative show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$ 5

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n,$$

where $|z| < \infty$. 5

(c) State and prove Laurent's theorem.

(d) (i) Using definition of derivative, show that $f(z) = |z|^2$ is nowhere differentiable except at $z = 0$. 5

(ii) Define singular points of a function. Determine singular points of the functions :

$$f(z) = \frac{2z+1}{z(z^2+1)} \ ;$$

$$g(z) = \frac{z^3+i}{z^2-3z+2}$$ 1+4=5

(e) (i) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D. Prove that the families of curves $u(x, y) = c_1$, $v(x, y) = c_2$ are orthogonal.

(ii) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that

$$|f(z)| \le M \text{ for all } z \text{ in } C$$

then show that

$$\left| \int_C f(z)\, dz \right| \le ML .$$ 5+5=10

(f) (i) Prove that two non-zero complex numbers z_1 and z_2 have the same moduli if and only if $z_1 = c_1 c_2$, $z_2 = c_1 \bar{c}_2$, for some complex numbers c_1, c_2. 4

(ii) Show that mean value theorem of integral calculus of real analysis does not hold for complex valued functions $w(t)$. 3

(iii) State Cauchy-Goursat theorem. 1

(iv) Show that $\lim\limits_{z \to \infty} \dfrac{z^2 + 1}{z - 1} = \infty$. 2

OPTION–B

(For Old Syllabus)

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Write the statement of the First Fundamental Theorem of Calculus.

   (b) Evaluate $\int_0^\infty e^{-x}\,dx$, if it exists.

   (c) Prove that $\Gamma(1) = 1$.

   (d) Define a complete metric space.

   (e) Describe an open ball in the discrete metric space (X, d).

   (f) $(A \cup B)^0$ need not be $A^0 \cup B^0$ — Justify it where A and B are subsets of a metric space (X, d).

   (g) Find the derived sets of the intervals $(0, 1)$ and $[0, 1]$.

(h) Let A and B be two subsets of a metric space (X, d). Which of the following is not correct?

(i) $A \subseteq B \Rightarrow A' \subseteq B'$

(ii) $(A \cap B)' \subseteq A' \cap B'$

(iii) $A' \cap B' \subseteq (A \cap B)'$

(iv) $(A \cup B)' = A' \cup B'$

(i) The Euclidean metric on $\mathbb{R}^n$ is defined as

(i) $d(x, y) = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{\frac{1}{2}}$

(ii) $d(x, y) = \left\{ \sum_{i=1}^{n} |x_i - y_i|^p \right\}^{\frac{1}{p}}$

where $p \geq 1$

(iii) $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$

(iv) $d(x, y) = \sup_{1 \leq i \leq n} |x_i - y_i|$

where $x = (x_1, x_2, \cdots x_n)$

$y = (y_1, y_2, \cdots y_n)$

are any two points in $\mathbb{R}^n$.

(Choose the correct answer)

(j) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be continuous on X. Then for any $B \subseteq Y$.

(i) $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$

(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

(iii) $\overline{f(B)} \subset f(\overline{B})$

(iv) $f(\overline{B}) \subset \overline{f(B)}$

(Choose the correct answer)

2. Answer the following questions : 2×5=10

(a) Let $f(x) = x$ on $[0, 1]$ and

$$P = \left\{ x_i = \frac{i}{4}, \ i = 0, 1, \cdots, 4 \right\}$$

Find $L(f, P)$ and $U(f, P)$.

(b) Let $f : [0, a] \to \mathbb{R}$ be given by $f(x) = x^2$. Find

$$\int_0^a f(x)\, dx$$

(c) Let (X, d) be a metric space and A, B be subsets of X. Prove that $(A \cap B)^0 = A^0 \cap B^0$.

(d) If A is a subset of a metric space (X, d), prove that $d(A) = d(\overline{A})$.

(e) Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that if a mapping $f : X \to Y$ is continuous on X, then $f^{-1}(G)$ is open in X for all open subsets G of X.

3. Answer ***any four*** parts : 5×4=20

(a) Prove that $f(x) = x^2$ on $[0, 1]$ is integrable.

(b) Show that $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r}{r^2 + n^2} = \log \sqrt{2}$

(c) Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all}$$

$x, y \in X$. Prove that d' is a metric on X.

(d) Let $X = c[a, b]$ and

$d(f, g) = sup\{|f(x) - g(x)| : a \leq x \leq b\}$

be the associated metric where $f, g \in X$. Prove that (X, d) is a complete metric space.

(e) Let (X, d) be a metric space. Prove that a finite union of closed sets is closed.
Infinite union of closed sets need not to closed — Justify it. 3+2=5

(f) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. If $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X, prove that $\{f(x_n)\}_{n \geq 1}$ is a Cauchy sequence in Y.

4. Answer ***any four*** parts : 10×4=40

(a) (i) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that f is integrable. 5

(ii) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p} dx$ for various values at p. 5

(b) (i) Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

Using it prove that for $-1 < a < 0$ and $n \in \mathbb{N}$,

$$S_n = \int_a^0 \frac{x^n}{1+x}\,dx \to 0 \text{ as } n \to \infty$$

3+2=5

(ii) Let $f:[a,b] \to \mathbb{R}$ be monotone. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(a)(c-a) + f(b)(b-c)$$

5

(c) (i) Prove that a convergent sequence in a metric space is a Cauchy sequence.
Show that in the discrete metric space every Cauchy sequence is convergent. 3+2=5

(ii) Define an open set in a metric space (X, d).
Prove that in any metric space (X, d), each open ball is an open set. 1+4=5

(d) (i) Let (X, d) be a metric space and F be a subset of X. Prove that F is closed in X if and only if F^c is open in X. 5

(ii) Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Prove that Z is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$. 5

(e) (i) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

(ii) Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y. 4

(f) (i) Show that the function $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x}$ is not uniformly continuous. 5

(ii) Let (X,d) be a metric space and let $x\in X$ and $A\subseteq X$ be non-empty. Prove that $x\in A$ if and only if $d(x,A)=0$. 5

(g) (i) Define a connected set in a metric space.
Prove that if Y is a connected set in a metric space (X,d), then any set Z such that $Y\subseteq Z\subseteq \overline{Y}$ is connected. 1+4=5

(ii) Let (X,d) be a metric space. Prove that the following statements are equivalent :

(a) (X,d) is disconnected

(b) there exists a continuous mapping of (X,d) onto the discrete two element space (X_0, d_0). 5

(h) Let $(\mathbb{R}, d)$ be the space of real numbers with the usual metric. Prove that a subset I of $\mathbb{R}$ is connected if and only if I is an interval.
