Total number of printed pages–16

3 (Sem–6/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6016

(New Syllabus/Old Syllabus)

Full Marks : 80/60

Time : Three hours

The figures in the margin indicate full marks for the questions.

New Syllabus

Full Marks : 80

(Riemann Integration and Metric Spaces)

1. Answer the following as directed : 1×10=10

   (a) Define the discrete metric d on a non-empty set X.

Contd.

(b) Let F_1 and F_2 be two subsets of a metric space (X, d). Then

(i) $\overline{F_1 \cup F_2} = \overline{F_1} \cap \overline{F_2}$

(ii) $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$

(iii) $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$

(iv) $\overline{F_1 \cap F_2} = \overline{F_1} \cup \overline{F_2}$

(Choose the correct option)

(c) Let (X, d) be a metric space and $A \subset X$. Then

(i) $Int\,A$ is the largest open set contained in A.

(ii) $Int\,A$ is the largest open set containing A.

(iii) $Int\,A$ is the intersection of all open sets contained in A.

(iv) $Int\,A = A$

(Choose the correct option)

(d) Let (X, d) be a disconnected metric space.

We have the statements :

I. There exists two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

II. There exists two non-empty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(i) Only I is true

(ii) Only II is true

(iii) Both I and II are true

(iv) None of I and II is true

(Choose the correct option)

(e) Find the limit points of the set of rational numbers Q in the usual metric R_u.

(f) In a metric space, the intersection of infinite number of open sets need not be open. Justify it with an example.

(g) Define a mapping $f : X \to Y$, so that the metric spaces $X = [0, 1]$ and $Y = [0, 2]$ with usual absolute value metric are homeomorphic.

(h) Define Riemann sum of f for the tagged partition (P, t).

(i) State the first fundamental theorem of calculus.

(j) Examine the existence of improper Riemann integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

2. Answer the following questions : 2×5=10

(a) Prove that in a metric space (X, d) every open ball is an open set.

(b) Prove that the function $f : [0,1] \to R$ defined by $f(x) = x^2$ is an uniformly continuous mapping.

(c) Let d_1 and d_2 be two matrices on a non-empty set X. Prove that they are equivalent if there exists a constant K such that

$$\frac{1}{K} d_2(x, y) \le d_1(x, y) \le K d_2(x, y)$$

(d) If m is a positive integer, prove that $\Gamma{m+1} = m!$

(e) Let $f(x) = x$ on $[0, 1]$.

Let $P = \left\{ x_i = \frac{i}{4},\ i = 0, \ldots 4 \right\}$

Find $L(f, P)$ and $U(f, P)$.

3. Answer the following questions **(any four)**: 5×4=20

(a) Let (X, d) be metric space and F be a subset of X. Prove the F is closed in X if and only if F^c is open.

(b) Define diameter of a non-empty bounded subset of a metric space (X, d). If A is a subset of a metric space (X, d), then prove that $d(A) = d(\overline{A})$. 1+4=5

(c) Let (X, d) be a metric space. Then prove that the following statements are equivalent :

(i) (X, d) is disconnected.

(ii) There exists two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(d) Let $f, g : [a, b] \to R$ be integrable functions. Then prove that $f+g$ is integrable and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(e) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p}\,dx$ for various values of p.

(f) Consider $f : [0, 1] \to R$ defined by $f(x) = x^2$. Prove that f is integrable.

4. Answer the following questions : 10×4=40

(a) (i) Let X be the set of all bounded sequences of numbers $\{x_i\}_{i \geq 1}$ such that $\sup_i |x_i| < \infty$.

For $x = \{x_i\}_{i \geq 1}$ and $y = \{y_i\}_{i \geq 1}$ in X define $d(x, y) = \sup_i |x_i - y_i|$.

Prove that d is a metric on X. 5

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify with an example. 4+1=5

Or

(a) (i) Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of reals. 4

(ii) Show that the metric space $X = \mathbb{R}^n$ with the metric given by $d_p(x, y) = \left(\Sigma |x_i - y_i|^p\right)^{1/p}, \quad p \geq 1$ where $x = (x_1, x_2 \ldots\ldots x_n)$ and $y = (y_1, y_2 \ldots\ldots y_n)$ are in $\mathbb{R}^n$ is a complete metric space. 6

(b) (i) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$. If f is continuous on X, prove the following : 3+3=6

(i) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all subsets of B of Y

(ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X

(ii) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. Prove that if $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X, then $\{f(x_n)\}_{n \geq 1}$ is a Cauchy sequence in Y. 4

Contd.

Or

(b) Define fixed point of a mapping $T : X \to X$. Let $T : X \to X$ be a contraction of the complete metric space (X, d). Prove that T has a unique fixed point. 2+8=10

(c) (i) Prove that if the metric space (X, d) is disconnected, then there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). 5

(ii) Let (X, d) be a metric space and A^0, B^0 are interiors of the subsets A and B respectively. Prove that

$$(A \cap B)^0 = A^0 \cap B^0;$$
$$(A \cup B)^0 \supseteq A^0 \cup B^0.$$ 5

Or

(c) (i) When is a non-empty subset Y of a metric space (X, d) said to be connected? Let (X, d_X) be a connected metric space and $f : (X, d_X) \to (Y, d_Y)$ be a continuous mapping. Prove that the space $f(X)$ with the metric induced from Y is connected. 5

(ii) Let (X, d) be a metric space and $Y \subseteq X$. If X is separable then prove that Y with the induced metric is also separable. 5

(d) (i) If f is Riemann integrable on $[a, b]$ then prove that it is bounded on $[a, b]$. 5

(ii) When is an improper Riemann integral said to exist? Show that the improper integral of $f(x) = |x|^{-1/2}$ exists on $[-1, 1]$ and its value is 4. 1+4=5

Or

(d) (i) Let $f : [a, b] \to R$ be integrable. Then prove that the indefinite integral $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$.

Further prove that if f is continuous at $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$. 3+3=6

(ii) Evaluate

$$\lim_{x \to \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{\sqrt{n^3}} = \frac{2}{3}$$ 4

Contd.