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3 (Sem–6/CBCS) MAT HC 1 (N/O)

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6016

New Syllabus

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

Old Syllabus

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

New Syllabus

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

1. Answer the following as directed :

$1 \times 10 = 10$

(a) Let $f:[a,b] \to R$ be a bounded function and P, Q are partitions of $[a,b]$. If Q is a refinement of P, then

(i) $L(f,Q) \le L(f,P)$

(ii) $U(f,P) \le U(f,Q)$

(iii) $U(f) \le L(f)$

(iv) $L(f) \le U(f)$

(Choose the correct option)

(b) Find the value of $\int_0^{\infty} e^{-x} dx$.

(c) Show that $\Gamma(1) = 1$.

(d) Define Cauchy sequence in a metric space.

(e) State whether the following statement is true **or** false :

"Each subset of a discrete metric space is open."

(f) If the mapping $d: R^2 \times R^2 \to R$ is defined as $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$, then which one of the following statements is true?

(i) d is the usual metric on R^2

(ii) d is uniform metric on R^2

(iii) d is a pseudo metric on R^2

(iv) None of the above statements is true

(g) Which of the following statements is not true?

(i) In a metric space countable union of open sets is open

(ii) In a metric space finite union of closed sets is closed

(iii) A non-empty subset of a metric space is closed if and only if its complement is open

(iv) None of the above statements is true

(h) When is a metric space said to be connected.

(i) State whether the following statement is true **or** false :

"Image of an open set under a continuous function is open."

(j) Under what condition the metric spaces (X, d_X) and (Y, d_Y) are said to be equivalent?

2. Answer the following questions : 2×5=10

(a) Let $u, v : [a, b] \to R$ be differentiable and u', v' are integrable on $[a, b]$. Then show that

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

(b) Show that a subset F of a metric space (X, d) is closed if and only if $\overline{F} = F$.

(c) Let (Y, d_Y) be a subspace of a metric space (X, d_X) and $S_X(z, r)$ and $S_Y(z, r)$ are open balls with center at $z \in Y$ and radius r in the metric space (X, d_X) and (Y, d_Y) respectively.

Prove that $S_Y(z, r) = S_X(z, r) \cap Y$.

(d) Show that the image of a Cauchy sequence under uniformly continuous function is again a Cauchy sequence.

(e) Show that a contraction mapping on a metric space is uniformly continuous.

3. Answer ***any four*** questions : 5×4=20

(a) Show that a bounded function $f:[a, b] \to R$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

(b) Let g be a continuous function on the closed interval $[a, b]$ and the function f be continuously differentiable on $[a, b]$. Further if f' does not change sign on $[a, b]$, then show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

(c) Let (X, d) be a metric space and the function $d^*: X \times X \to R$ is defined as

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$$

Show that (X, d^*) is a bounded metric space.

(d) Let Y be a non-empty subset of the metric space (X, d). Prove that the subspace (Y, d_Y) is complete if and only if Y is closed on (X, d).

(e) Show that composition of two uniformly continuous functions is also uniformly continuous.

(f) Show that a metric space (X, d) is disconnected if and only if there exists a continuous function of (X, d) onto the discrete two element space (X_0, d_0), i.e., $X_0 = \{0, 1\}$ and d_0 is the discrete metric on X_0.

4. Answer the following questions : 10×4=40

(a) Let f be a function on an interval J with nth derivative $f^{(n)}$ continuous on J. If $a, b \in J$, then show that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(b-a)^{n-1} + R_n$$

where, $R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$

Or

Let $f:[0,1] \to R$ be continuous and $c_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $n \in N$.

Then show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(c_i) = \int_0^1 f(x)\,dx .$$

Hence show that

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{r^2+n^2} = \log\sqrt{2} .$$ 5+5=10

(b) Let l_p $(p \geq 1)$ be the set of all sequences of real numbers such that if $x = \{x_n\}_{n \geq 1} \in l_p$, then $\sum_{i=1}^{\infty} |x_i|^p < \infty$.

Prove that the function $d : l_p \times l_p \to R$ defined by $d(x, y) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^p \right\}^{\frac{1}{p}}$ is a metric on l_p. Also show that l_p is a complete metric space. 4+6=10

Or

(i) Let (X, d) be a metric space and $\{x_n\}_{n \geq 1}$, $\{y_n\}_{n \geq 1}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then show that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. 4

(ii) Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Then show that Z is closed in Y if and only if there exists a closed set $F \subseteq X$ such that $Z = F \cap Y$. 6

(c) What are meant by contraction mapping and fixed point of a contraction mapping in a metric space? If $T: X \to X$ is a contraction mapping on a complete metric space, then show that T has a unique fixed point. (1+1)+8=10

Or

If (X, d) be a metric space, then show that the following statements are equivalent :

(i) (X, d) is disconnected.

(ii) There exist two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(iii) There exist two non-empty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(iv) There exists a proper subset of X that is both open and closed in X.

(d) (i) Let $f:[a, b] \rightarrow R$ be integrable and $F(x)=\int_a^x f(t)\,dt$; $x \in [a, b]$. Show that F is continuous on $[a, b]$. Also show that F is differentiable at $x \in [a, b]$ if f is continuous at $x \in [a, b]$ and $F'(x)=f(x)$. 7

(ii) Let (X, d) be a metric space and $\rho: X \times X \rightarrow R$ be defined by

$$\rho(x, y)=\frac{d(x, y)}{1+d(x, y)} \quad ; \quad x, y \in X .$$

Show that d and ρ are equivalent metrics. 3

Or

(iii) Show that a subset G of a metric space (X, d) is open if and only if it is the union of all open balls contained in G. 5

(iv) Give example, with justification, of a homeomorphism from a metric space onto another metric space which is not an isometry. 5

Old Syllabus

Full Marks : 60

(Complex Analysis)

Time : Three hours

1. Answer the following as directed : 1×7=7

   (a) Determine the accumulation point of the set $z_n = \frac{i}{n}\,(n = 1, 2, 3, \cdots)$

   (b) Describe the domain of $f(z) = \frac{z}{z + \bar{z}}$.

   (c) Define an entire function.

   (d) Determine the singular points of
   $$f(z) = \frac{2z+1}{z\left(z^2+1\right)}$$

   (e) The value of $\log e$ is

   (i) 1

   (ii) $1 + 2n\pi i$

   (iii) $2n\pi i$

   (iv) 0

   (Choose the correct option)

(f) $\lim_{n \to \infty} \left(-2 + i \frac{(-1)^n}{n^2} \right)$ is equal to

(i) 0 (ii) -2

(iii) $-2+i$ (iv) limit does not exist

(Choose the correct option)

(g) The power expression for $\cos z$ is

(i) $\frac{e^z + e^{-z}}{2}$ (ii) $\frac{e^{iz} + e^{-iz}}{2}$

(iii) $\frac{e^{iz} + e^{-iz}}{2i}$ (iv) $\frac{e^z - e^{-z}}{2}$

(Choose the correct option)

2. Answer the following questions : 2×4=8

(a) Sketch the set

$$|z - 1 + i| \leq 1$$

(b) Prove that $f'(z)$ exists every where for the function $f(z) = iz + 2$.

(c) If $f(z) = \frac{z}{\bar{z}}$, prove that $\lim_{z \to 0} f(z)$ does not exist.

(d) Evaluate $\int_1^2 \left(\frac{1}{t} - i \right)^2 dt$.

3. Answer **any three** questions from the following: 5×3=15

(a) (i) Show that if e^z is real, then

$Im\, z = n\pi \; (n = 0, \pm 1, \pm 2, \cdots)$ 3

(ii) Show that $exp(2 \pm 3\pi i) = -e^2$. 2

(b) Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then prove that

$$\lim_{z \to z_0} f(z) = w_0 \text{ if}$$

$$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and}$$

$$\lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0$$

(c) Show that $f'(z)$ exists everywhere, when $f(z) = e^z$.

(d) Evaluate $\int_C \frac{dz}{z}$, where C is the top half of the circle $|z| = 1$ from $z = 1$ to $z = -1$.

(e) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Applying Cauchy's integral formula, evaluate $\int_C \frac{e^{-z} dz}{z - \left(\pi i/2\right)}$.

4. Answer **either** (a) **or** (b) and (c):

(a) Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Prove that the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ there.

Also show that $f'(z) = u_x + iv_x = v_y - iu_y$ where partial derivatives are to be evaluated at (x_0, y_0). 10

Or

(b) If z_0 and w_0 are points in the z-plane and w-plane respectively, then prove that $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$

Hence show that $\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty$

4+2=6

(c) If $w = f(z) = \bar{z}$, examine whether $\frac{dw}{dz}$ exists or not. 4

5. Answer **either** (a) **or** (b) :

(a) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that $|f(z)| \leq M$ for all points z on C at which $f(z)$ is defined, then prove that

$$\left| \int_C f(z)\,dz \right| \leq ML$$

Hence show that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \pi/3,$$ where C is the arc of the semicircle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. 10

Or

(b) State and prove Liouville's theorem.

6. Answer **either** *(a)*, *(b)*, *(c)* **or** *(d)*:

(a) Prove that if a series of complex numbers converges, then the nth term converges to zero as n tends to infinity. 3

(b) Test the convergency of the sequence

$$z_n = \frac{1}{n^3} + i \; (n = 1, 2, \cdots)$$ 3

(c) Find Maclaurin's series for the entire function $f(z) = l^z$. 4

Or

(d) Suppose that a function f is analytic throughout a disc $|z - z_0| < R_0$ centred at z_0 and with radius R_0. Then prove that $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)$$

where $a_n = \dfrac{f^n(z_0)}{n!} \; (n = 0, 1, 2, \cdots)$
